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Three Dimensional Gauge Theories and Monopoles

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Abstract

The coulomb branch of $N = 4$ supersymmetric Yang-Mills gauge theories in $d = 2 + 1$ is studied. A direct connection between gauge theories and monopole moduli spaces is presented. It is proposed that the hyper-Kähler metric of supersymmetric $N = 4$ $SU(N)$ Yang-Mills theory is given by the charge N centered moduli space of BPS monopoles in $SU(2)$. The theory is compared to $N = 2$ supersymmetric Yang-Mills theory in four dimensions through compactification on a circle of the latter. It is found that rational maps are appropriate to this comparison. A BPS mass formula is also written for particles in three dimensions and strings in four dimensions.

1 Introduction

Recently there has been some interest in the dynamics of gauge theories in three dimensions. In [1] the dynamics of $U(1)$ and $SU(2)$ gauge theories was derived from string theory by looking at compactification of type I theory on T^3. This work was motivated by an interesting idea that branes can probe the geometry of space time [2] and was applied to four dimensions in [3] following the work of [4] on F-theory and orientifolds. In [5] a detailed discussion of the dynamics of $U(1)$ and $SU(2)$ gauge theories and comparison to expectations from string theory was performed. It was found that the Coulomb branch of the moduli space for low number of flavors is given by the moduli space of two monopoles. The aim of this paper is to generalize some of the results to $SU(N)$ Yang-Mills theories. We will make a direct connection with the moduli space of charge N $SU(2)$ monopoles.

In a recent work [6] we looked at an example of two intersecting five branes in M-theory which intersect over a 3-brane. The theory on the intersection is associated with the appearance of tensionless strings in four dimensions. The theory has two phases which are connected by the tensionless string point. One in which the branes are on top of each other and it was argued in [6] that they form a bound state. Further interpretation of this phase is still lacking. It was also argued that at the point of intersection a $U(1)$ gauge field and a hypermultiplet are unHiggsed and thus possibly the theory on the intersection has a Coulomb phase corresponding to the $U(1)$ field. The other phase is when the two branes are separated. This phase has a moduli space of vacua which is parameterized by a linear (tensor) multiplet in four dimensions. Further reduction of the two five brane configuration to two four branes intersecting over a two brane leads to the study of a gauge theory in three dimensions. There appear to be many similarities between the linear multiplet moduli space in four dimensions and the Coulomb branch of the three dimensional $U(1)$ gauge theory. This similarity has presumably origins in string theory. In particular the moduli space of those theories looks like a two monopole moduli space. So a natural question is to ask do we see the monopoles in the string theory? A partial answer can be given by viewing the two five branes which lie in say 12345 and 12367 directions as two magnetic monopoles which move in the 8,9,10 directions. Compactification of the 3-rd direction which leads from the four dimensional theory to the three dimensional theory then provides the reason why the two moduli spaces are similar. The above observation also suggests that the Coulomb branch of $N = 4$ gauge theories in three dimensions is connected to the moduli space of n monopoles where n is related to the rank of the gauge group.

The aim of this paper is to study the relation of $SU(N_c)$ gauge theories in three

dimensions to a system of N_c monopoles. In particular we find that semiclassically the Coulomb branch of $SU(N_c)$ gauge theory is the moduli space of N_c monopoles when the monopoles have large separations. Moreover we identify the moduli space of $SU(N)$ Yang-Mills with the moduli space of N monopoles.

The outline of the paper goes as follows. In section 2 we describe the three dimensional models and calculate the semiclassical metric. We also relate them to gauge theories on $\mathbb{R}^3 \times S^1$ with radius R. We discuss the structure of the moduli spaces and make the comparison to the moduli space of $SU(2)$ monopoles. In section 3 we discuss the R-dependence of the metric. We derive a formula for large R which relates the period matrix (matrix of couplings and theta angles) associated to four dimensional gauge theories with $N = 2$ supersymmetry to the Dirac connection on the moduli space of monopoles. In section 4 we review the correspondence of monopoles with the rational maps and discuss their application to three dimesional gauge theories. The picture associated with rational maps has a very natural role in connecting the three dimensional $N = 4$ gauge theories to the four dimensional $N = 2$ gauge theories. We discuss their implications. We give a few examples and introduce a BPS mass formula in three dimensions. This formula is also related to a BPS tension of a string in four dimensions.

While writing this paper two papers appeared [17, 18] which discuss problems related to the problems discussed in this paper.

2 $N = 4$ SU(N) gauge theories in three dimensions

Three dimensional gauge theories with $N = 4$ supersymmetry have three types of global symmetries. It is convenient to perform a dimensional reduction from $N = 1$ supersymmetric gauge theories in six dimensions in order to understand the symmetries. There is an $SU(2)_R$ symmetry which acts on the six dimensional fermions. The fermions and supercharges transform as doublets of this symmetry. The vector multiplet in three dimesions contains three scalars ϕ_i, $i = 1, 2, 3$. There is a rotation group under which these scalars transform as a vector. We will denote its double cover $SU(2)_N$. The last symmetry is the rotation group of three dimensional Euclidean space $\mathbb{R}^3$. Its double cover will be denoted $SU(2)_E$.

The matter content of a $N = 4$ super Yang-Mills theory consists of a vector multiplet which transforms in the adjoint representation of the gauge group G. We will be interested in this paper in studying the Coulomb branch in the case where the gauge group is $SU(N_c)$. Under $SU(2)_R \times SU(2)_N \times SU(2)_E$ the fermions of a vector multiplet transform as $(\mathbf{2},\mathbf{2},\mathbf{2})$

and the scalars transform as (**1**,**3**,**1**).

The potential energy for the scalars is

$$V = \frac{1}{e^2}\sum_{i<j} Tr[\phi_i,\phi_j]^2, \tag{2.1}$$

where e is the gauge coupling. The potential V vanishes if the ϕ_i commute; the space of zeros of V has dimension $3r$ where r is the rank of the gauge group. At a generic point in moduli space the gauge group is broken to $U(1)^r$. In addition to the $3r$ scalars we have also r massless photons which are each superpartners of the three scalars. Those photons are dual to compact scalars so the moduli space of the Coulomb branch is parameterized by $4r$ scalars. The low-energy effective action is a linear sigma model with target space a hyper-Kähler manifold of quaternionic dimension r. In addition to the vector multiplets we will at times couple the theory to N_f hypermultiplets which transform in the fundamental representation of the gauge group.

The low-energy (leading) component of the N=2 theory in d=3+1 dimensions is written in terms of a chiral and vector $N=1$ superfield ϕ^a and W^a_α. The low-energy theory before compactification is

$$S = \Im \int d^4x d^2\theta \frac{1}{2}\mathcal{F}_{ab}(\phi) W^{a\alpha}W^b_\alpha + \Im \int d^4x d^2\theta d^2\bar\theta \mathcal{F}_a(\phi)\bar\phi^a \ . \tag{2.2}$$

We only consider the fields living in the Cartan sub-algebra of the bosonic sector, in which the action is given by

$$S_g = \int d^4x \frac{1}{4e^2_{ij}} F^i_{\mu\nu}F^{j,\mu\nu} + \frac{i\theta_{ij}}{32\pi^2}F^i_{\mu\nu}\tilde F^{j,\mu\nu} + \int d^4x \frac{1}{2e^2_{ij}}\partial_\mu\bar\phi^i\partial^\mu\phi^j \ . \tag{2.3}$$

We have kept only terms arising in the action which have two derivatives; the terms in $(D_\mu\phi)^{i,\dagger}(D^\mu\phi)^j$ other than the kinetic one do not contribute to the low-energy effective action of the scalars.

The effective three dimensional theory is found by compactifying on a spacetime $R^3\times S^1$ and keeping only the r photons and the r scalars arising from the fourth component of the 4-dim gauge fields. We further will denote the complex scalars in the chiral $N=1$ multiplet by the real pair $\phi^i = \phi^i_1 + i\phi^i_2$.

In the following we give the description of the moduli space of the three-dimensional gauge theory $SU(N_c)$ in the semi-classical regime. The space of fields ϕ_i which minimize the scalar potential in eq.(2.1) can be parameterized in the Cartan sub-algebra as

$$\phi_i = \mathrm{diag}\left[x^i_1, ..., x^i_{N_c}\right] \tag{2.4}$$

with the condition that $\sum_{j=1}^{N_c} x_{ij} = 0, i = 1,2,3$. This parameterization is up to a Weyl transformation and so the space of zeroes of V is given by $\mathbb{R}^{3N_c}/S_{N_c}$ restricted to the above condition. The classical moduli space of the scalar fields, however, is required not to have any points of enhanced gauge symmetry. These points exist whenever the discriminant

$$\Delta = \prod_{i \neq j} |\vec{\phi}_i - \vec{\phi}_j| \tag{2.5}$$

vanishes. In the quantum theory we expect the metric at these points to be smooth or singular depending on the number of flavors Semi-classically we delete the points in which at least two of the scalar field expectation values are the same. There are $\frac{1}{2}N_c(N_c-1)$ surfaces Δ_{ij} defined by the condition on the pairs $\vec{\phi}_i = \vec{\phi}_j$ in which we delete. The classical moduli space of a completely broken gauge theory is then $\mathcal{M}_{cl} = \left(\mathbb{R}^{3N_c} - \{\Delta_{ij}\}\right)/S_{N_c}$. S_{N_c} is the permutation group for N_c elements. Under the action of an element P_{ij} which exchanges the fields $\vec{\phi}_i \leftrightarrow \vec{\phi}_j$ the corresponding deleted surfaces in $\mathbb{R}^{3N_c}$ are identified. There are points of enhanced gauge symmetry whenever scalar expectation values meet; for example, points in which M scalar vevs are equal give rise to a symmetry group $SU(M) \times U(1)^{N_c-M}$ although various product groups are possible.

The Weyl group also acts on the compact scalars dual to the photons $\theta_i, i = 1, \ldots N_c$, $\sum_{i=1}^{N_c} \theta_i = 0$ by an exchange in S_{N_c}. Including these scalars the classical $4N_c$-dimensional moduli space of completely broken vacua is

$$\mathcal{M}_N^{\mathrm{cl}} = \frac{(\mathbb{R}^{3N_c} - \Delta_{ij}\}) \times T^{N_c}}{S_{N_c}}. \tag{2.6}$$

One-loop corrections generate a non-trivial Dirac connection on the T^{N_c} given by $d\theta_i + \Gamma_{ij}^k dx_k^j$, as described below, and the T_{N_c} is promoted to a non-trivial fibre. Classically this vanishes and the vacua is simply a direct product of the compact scalars on T^{N_c} together with the scalars on R^{3N_c}.

The classification of the possible T^{N_c} fibre bundles with the appropriate symmetry under the permutation group is a generalization of the results in [5]. The base, $\mathbb{R}^{3N_c} - \Delta_{ij}$, in eq.(2.6) has a second homology group $H_2(\mathcal{M}_{N_c}^{\mathrm{cl}}, Z)$ with dimension $\frac{N_c(N_c-1)}{2}$. A homology basis is given by the 2-spheres S_{ij}^2 defined by $|\vec{x}_i - \vec{x}_j| = r_{ij} = \text{const}$, $x_k = \text{const}$, $i \neq j \neq k$. In order to determine the fiber we must specify the connection one form associated to the i-th generator of T^{N_c} around the different homology generators. However, examining the connections restricted to each of the S_{ij}^2 together with the permutation symmetry reduces it to the standard Dirac monopole connection over S^2 with a magnetic charge s; the action of the permutation group implies the topological index is the same around

each of the spheres.

In the quantum gauge theory the integer s which labels the non-trivial semi-classical T^{N_c} fiber at infinity may be found either by a one loop computation or by a counting of the fermion zero modes which give rise to an instanton correction. It is noteworthy that in three dimensions magnetic monopoles in an unbroken $U(1)$ subgroup of the gauge group may appear as instantons. It was found [5] to be $s = 4$ for an $SU(2)$ group by analyzing the symmetries of the instanton. This analysis is independent of the gauge group. Furthermore by counting the zero modes of a vector multiplet in the presense of one monopole [7] it is found to be the same for arbitrary gauge group.

$$s = 4. \tag{2.7}$$

Due to such an instanton the $SU(2)_N$ symmetry is broken down to $U(1)_N$ by the expectation value of the scalars. Under this symmetry there are 4 zero modes with charge $\frac{1}{2}$ which come from the gauge multiplet. An r-instanton contribution to the metric will then be possible if

$$4r = 4. \tag{2.8}$$

This implies that $r = 1$. There can be also contributions from anti-monopoles which will have zero modes with opposite $U(1)_N$ charges. The counting then does not change essentially, r is replaced with the net instanton number. Note that this counting implies that the value for the coefficient of the Dirac potential in the semi-classical result above to be $s = 4$. Indeed, the asymptotic form of the metric for a charge k monopole has a coefficient of the Dirac potential which is independent of k [8]. Thus, the asymptotic forms of the monopole and gauge theory metrics coincide and lends further support to the identification.

The classical vacuum consisting of the $3N_c$ scalars and N_c compact duals to the photon has a flat metric described by

$$ds^2 = \sum_{i=1}^{N_c} (\frac{1}{e^2} d\vec{x}_i^2 + e^2 d\theta_i^2). \tag{2.9}$$

where the Weyl group action is implied. As was found in [1] one loop corrections change the metric in the asymptotic regime. Considering a $U(1)$ gauge group the one-loop effect gives rise to a non-trivial connection on the periodic scalar in T and is

$$d\theta \to d\theta - s\vec{w} \cdot d\vec{x}. \tag{2.10}$$

The vector $\vec{w}(\vec{x})$ satisfies the field equation of a Dirac point monopole

$$\nabla \times \vec{w} = \nabla(\frac{1}{|\vec{x}|}). \tag{2.11}$$

$N = 4$ supersymmetry requires the metric to be hyper-Kähler. Necessary conditions for the metric to be hyper-Kähler * then lead to the corresponding correction to the metric on R^3 given by

$$\frac{1}{e^2} \to \frac{1}{e^2} - \frac{s}{|\vec{x}|}. \tag{2.12}$$

The metric is semi-classically corrected to be of the Taub-Nut type with a negative mass parameter for positive s. It possesses a real singularity, i.e. it is not geodesically complete, when the latter side of equation (2.12) is zero; this occurs for positive s.

Note that in the limit of large fields the semi-classical correction may be neglected and the metric reduces to the flat one given in [1]. The total correction can also be computed by a one loop calculation as it has power of e^0. Dimensional reasoning leads to the form of this correction and the coefficient s is the analogue of the beta function in four dimensions (it is proportional to the corresponding zero mode counting in $d = 4$).

We now generalize the semi-classical results to other examples, including $SU(N_c)$. We begin with the $U(1)$ gauge theory $s = -N_f$, where N_f is the number of hypermultiplets [5]. For $U(2)$ the asymptotic metric has the form

$$ds^2 = G d\vec{x}_-^2 + d\vec{x}_+^2 + G^{-1} d\tilde{\theta}_-^2 + d\theta_+^2, \qquad G = \frac{1}{e^2} - \frac{s}{r_{12}}, \tag{2.13}$$

with the following notations

$$\vec{x}_\pm = \frac{1}{\sqrt{2}}(\vec{x}_1 \pm \vec{x}_2), \qquad \theta_\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm \theta_2), \tag{2.14}$$

$$r_{12} = |\vec{x}_1 - \vec{x}_2|, \qquad \vec{w}_{12} = \vec{w}(\vec{x}_1 - \vec{x}_2), \tag{2.15}$$

$$d\tilde{\theta}_- = d\theta_- - s\vec{w}_{12} \cdot d\vec{x}_-. \tag{2.16}$$

The coordinates with subscript + represent the $U(1)$ fields which decouple from the $SU(2)$ part. For a configuration of two monopoles this corresponds to a decoupling of the center of mass motion. It has been proven in [5] that there are no further instanton corrections to the semi-classical results.

We generalize the semi-classical results for $SU(N_c)$ groups, with a $4(N_c - 1)$ dimensional moduli space, by generalizing the one-loop calculation and writing equation (2.13) in the form

$$ds^2 = g_{ij} d\vec{x}_i \cdot d\vec{x}_j + (g^{-1})_{ij} d\tilde{\theta}_i d\tilde{\theta}_j, \tag{2.17}$$

$$d\tilde{\theta}_i = d\theta_i - \frac{s}{2} \sum_{j=1}^{N_c} \vec{W}_{ij} \cdot d\vec{x}_j. \tag{2.18}$$

*see [8] and references therein.

The function $\vec{W}_{ij}$ is the potential arising from a Dirac point monopole at i at the point j. As such, we have a non-trivial connection on the T^{N_c} fibers given by

$$W_{ii} = -\sum_{i\neq j} w_{ij}, \qquad W_{ij} = w_{ij}, \qquad i \neq j. \tag{2.19}$$

Given the form of $\vec{W}_{ij}$, the hyper-Kähler condition for the metric in equation (2.17) as discussed in equation (2.23) enforces the remaining components of the metric to change to

$$g_{ii} = \frac{1}{e^2} - \frac{s}{2}\sum_{i\neq j}\frac{1}{r_{ij}} \qquad g_{ij} = \frac{s}{2}\frac{1}{r_{ij}}, \qquad i \neq j. \tag{2.20}$$

The metric (2.13) for $U(2)$ now takes the form given by equations (2.17), (2.18) and

$$g = \begin{pmatrix} \frac{1}{e^2} - \frac{s}{2r_{12}} & \frac{s}{2r_{12}} \\ \frac{s}{2r_{12}} & \frac{1}{e^2} - \frac{s}{2r_{12}} \end{pmatrix}, \qquad \vec{W} = \vec{w}_{12}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.21}$$

Viewing each pair of distinct indices i, j as defining an $SU(2)$ subgroup we can replace the indices $\{1,2\}$ with $\{i,j\}$ and get the form of equations (2.20) verifying the metric for $SU(N_c)$.

The three Kähler forms ω^a associated with the above semi-classical metrics have a simple explicit description:

$$\omega^a = -\frac{1}{2}g_{ij}\epsilon^{abc}dx_i^b \wedge dx_j^c + (d\theta_i - \frac{s}{2}\vec{W}_{ij}\cdot d\vec{x}_j^a) \wedge dx_i^a \ . \tag{2.22}$$

Under the $SO(3)$ rotational group these forms transform as a vector. Furthermore, in the absence of instanton corrections (plausibly in cases $N_f > N_c$ in $SU(N_c)$) the Kähler forms in eq.(2.22) are exact in the quantum theory.

The metrics in eq.(2.17) are special in that they possess N_c isometries generated by constant translations of the N_c coordinates θ_i. These isometries preserve all three Kähler forms ω_i of the hyperkähler metric (i.e. tri-holomorphic). Physically they correspond to the independent conservation of the $U(1)^N$ factors living in the Cartan sub-algebra of the gauge group. (Examples of the complete forms of these metrics have recently been used to describe moduli spaces of distinct fundamental monopoles in higher-rank gauge groups [10]). Furthermore, a $4N$ dimensional metric with the maximal number of tri-holomorphic isometries, i.e. N, may always be written in the form in eqs.(2.17) and (2.18) with

$$\frac{s}{2}(\partial_i^{(a)}W_{jk}^b - \partial_j^{(b)}W_{ik}^{(a)}) = \epsilon^{abc}\partial_i^{(c)}g_{jk} \qquad \partial_i^{(a)}g_{jk} = \partial_j^{(a)}g_{ik}\ , \tag{2.23}$$

where $\partial_i^{(a)} \equiv \partial/\partial x_i^a$ [20]. One may verify that the solution to the metric above satisfies these conditions. We do not expect this relation to hold for $N_f < N_c$ flavors, as we expect

non-perturbative instanton corrections to generate mass terms for the $U(1)$ gauge fields [9], thus breaking the isometries.

It is also useful as a consistency check to examine the limit to partial symmetry breaking of the semi-classical results; in the solution (2.17) we can test the breaking of the $SU(N_c)$ theory down to $SU(N_c-1)\times U(1)$ by setting N_c-1 of the fields to have equal expectation values. Explicitly we set

$$\vec{x}_i = \vec{x} + \vec{y}_i, \quad i = 1, \dots, N_c - 1, \qquad \vec{x}_{N_c} = (1 - N_c)\vec{x}, \tag{2.24}$$

where y_i are very small compared to $\vec{x}$ and satisfy $\sum_{i=1}^{N_c-1} \vec{y}_i = 0$. The metric then has terms which for large x goes to zero as $\frac{1}{|\vec{x}|}$. Neglecting this term produces the expected result that the x_{N_c} field decouples to a $U(1)$ field and the rest form the metric for the $SU(N_c-1)$ theory. At an enhanced $SU(2)$ subgroup, namely when two coordinates x_i, x_j coincide while the other are far away, we expect to get the structure as in equation (2.21), which is also verified.

Note that for positive s and sufficiently small fields the metric is no longer positive definite. The semiclassical approximation breaks down in this regime. However, it is known that there are further contributions to the metric which are exponentially small when the fields are large; they arise from monopole/instanton corrections in $d = 2+1$ and cure this problem which is seen asymptotically.

Monopole Comparison

The low-energy dynamics of a charge k monopole in an $SU(2)$ theory broken down to $U(1)$ is modeled by geodesic motion on a hyperKähler manifold of dimension $4k$. The metric factorizes for general k into the structure

$$M_k = \frac{\tilde{M}_k^0 \times S^1}{Z_k} \times \mathbb{R}^3 \tag{2.25}$$

The $\mathbb{R}^3$ and the S^1 are related to the physical decoupling of the overall center of mass and phase of the k-monopole. The k-fold cover of the moduli space breaks isometrically into a direct product of the form $\tilde{M}_k = \tilde{M}_k^0 \times S^1 \times \mathbb{R}^3$. In the asymptotic regime of moduli space where the k monopole may be thought of as separated charge one solitons, the k-fold cover of the metric asymptotically approaches the form

$$\tilde{M}_k \to (\mathbb{R}^3 \times S^1)^k \tag{2.26}$$

in which the parameters roughly label the positions and (asymptotically independent) $U(1)$ phases of the k monopoles possessing magnetic charge one. Furthermore, the reduced

(or centered) moduli space of a charge k monopole is of quaternionic dimension $k-1$ and is denoted by M_k^0; the k-fold cover is $\tilde{M}_k^0$.

On M_k the translations in $\mathbb{R}^3$ and rotations of $SO(3)$ act as isometries. Furthermore, the $SO(3)$ action rotates the three complex structures ω_i on the hyperkähler manifold M_k into eachother. We briefly discuss the Z_k action. The periodic S^1 coordinate ψ is associated with the unbroken $U(1)$ of the monopole in the sense of being a conjugate "momentum" coordinate to the electric charge. The Z_k group acts on this coordinate by $\psi \to \psi + \frac{2\pi}{k}$ together with a non-trivial action on the covering $\tilde{M}_k^0$. For the four-dimensional centered moduli space, M_2^0, the constraint of having an $SO(3)$ group of isometries which rotates the complex structures constrains the form to be uniquely determined.

The one loop corrected metric (2.19) coincides with the asymptotic metric for N_c monopoles which are located far apart from each other [8]. The three real scalar expectation values of the gauge theory correspond to the asymptotic positions of the monopoles. The scalar dual to the photon is identified with the large gauge transformation corresponding to each of the monopoles. The asymptotic topology of the moduli space M_k is of the same structure as the corresponding gauge theory in three dimensions. This correspondence goes further and we discuss it in the next sections.

3 R-Dependence

In this section we discuss the gauge theory on a four dimensional manifold given by $\mathbb{R}^3 \times S^1$ where the S^1 has radius R. We expect to find the known $N=2$ $d=4$ results in the limit $R \to \infty$ while the $R \to 0$ limit should produce the three dimensional gauge theories discussed in this paper. The scalars in the theory are the fourth components of the gauge fields, the dual photons, and the complex superpartners of the gauge fields. For a gauge group with rank r they parameterize locally $T^{2r} \times \mathbb{C}^r$.

The $SO(3)_N$ symmetry which acts on the scalars in the $R=0$ limit is broken down to $SO(2)_N$ for $R \to \infty$, which is usually called $U(1)_R$ in $d=4$. There are a S^2 worth of different ways in comparing the compactifications of the $N=1$ theory in six dimensions down to $d=3$ and $d=4$. We dimensionally reduce on the coordinates x^4, x^5, x^6 in compatifying to the $N=4$ three-dimensional theory, but in arriving to the $N=2$ four-dimensional result one may dimensionally reduce on any plane orthogonal to a unit vector in S^2. There is a natural correspondence between rational maps and monopoles which involves choosing a preferred direction. For this reason it is very convenient to look at

rational maps in order to study the R dependence of the metric on the moduli space and also to relate the two extreme limits of zero and infinite radius. We will introduce the rational maps in the next section and in this section we discuss the R-dependence of the theories.

Performing a duality transformation on the photons similar to that in [5] we obtain the low-energy theory on $\mathbb{R}^3 \times S^1$

$$L = \frac{1}{\pi R e_{ij}^2} db_i db_j + \frac{e_{ij}^2}{\pi R (8\pi)^2} d\tilde{\sigma}^i d\tilde{\sigma}^j + \frac{\pi R}{e_{ij}^2} \partial_\mu \bar{\phi}^i \partial^\mu \phi^j \tag{3.1}$$

with

$$d\tilde{\sigma}^i = d\sigma^i - \frac{\theta^{ij}}{\pi} db_j \ . \tag{3.2}$$

We can now compare the form of the metric induced by this action to the form of the semi-classical metric calculated in section 2.

As was explained in [5], there exist a distinguished complex structure ω in which there is no R dependence. This essentially follows from the holomorphy of the complex structure. We associate this particular combination of the three complex structures in the $N = 4$ model (at $R = 0$) with the choice of x^4 being the direction of the compactification on S^1, which corresponds to the third direction of the vector multiplet of $SO(3)$ of scalars in the notation of section two. We identify this prefered complex structure for all R as arising from the three-dimensional theory with the natural one arising on the rational maps defined along the preferred direction. This allows us to identify the fields b_i with the fields in the third direction x_i^3. It also allows us to identify the matrix of theta angles θ_{ij} as the Dirac connection in the third direction W_{ij}^3 and the matrix of coupling constants e_{ij}^{-2} as the metric on the moduli space g_{ij}.

In the theories that do not possess any instanton corrections for any R, we can use the hyperKähler constraints, equation (2.23) to derive a condition on the Dirac connection of the theory to the period matrix of couplings and theta angles. We set $W_{ij} = W_{ij}^1 - iW_{ij}^2$ and $z_j = x_j^1 + ix_j^2$ and the constraint takes the form

$$\partial_{z_i} \tau_{jk} = \partial_j^3 W_{ik} \tag{3.3}$$

This relation is an exact statement for these theories between $R = 0$ and $R = \infty$.

In cases where instantons correct the metric there is a breaking of the $U(1)$ symmetries and the semi-classical form of the metric on the moduli space, given in section two, no longer holds. It would be interesting to derive a similar relation for such cases, although in general the metrics possessing instanton corrections do not have the appropriate $U(1)$ isometries to be obtained by the condition in eq.(2.23).

In the dualized theory the structure of the classical moduli space is given by a N fibration over C^r with an R dependence. This is seen in the semi-classical form of the metric in eq.(3.2). In the case of $SU(2)$, for example, the structure of the quantum vacua is an elliptic fibration over C^2 with the area of the torus fiber $V_E = 1/16\pi R$ (in units where $\tau = \frac{\theta}{\pi} + \frac{8\pi}{e^2}$) and in the normalization of eq.(3.2). Semi-classically, the periodicity of the coordinates b_i and σ_i for all $SU(N_c)$ gauge groups together with the form of the metric in eq.(3.2) gives the corresponding structure of N as a T^{2r} bundle over C^r. The corresponding volume of N for higher rank gauge groups goes accordingly as $V_N \sim 1/R^r$ and diverges as $R \to 0$.

4 Rational Maps

We would like to consider the structure of the quantum moduli spaces of vacua for $N_f = 0$ in $d = 2+1$ through a generalization of the $SU(2)$ result. There a curve was introduced describing the (non-compact) Atiyah-Hitchin manifold, on which the physical $U(1)_R$ symmetry has a simple action. The analog for higher rank gauge groups is to consider a curve describing the non-compact moduli space. There is to every monopole configuration an associated space, the rational mappings from $CP^1 \to CP^1$ together with a constraint of a determinant equation; in this approach of looking at the moduli space the $U(1)_R$ symmetry of the vacuum is easily seen and the structure of the $R > 0$ of the compactification scale on S^1 is more clear.

A based rational map in $R_k(z)$ describing a charge k monopole is defined as

$$S(z) = \frac{p(z)}{q(z)} = \frac{\sum_{j=0}^{k-1} a_j z^j}{z^k + \sum_{j=0}^{k-1} b_{k-j} z^j} \ . \tag{4.1}$$

A theorem of Donaldson [11] states that there is a one-to-one correspondence between the universal cover of a k-monopole configuration (in SU(2)) and a map in R_k. Note that there are k complex solutions β_i to $q(z) = 0$ and another k values from the numerator $p(\beta_i)$. The rational map contains the $4k$ real dimensional space describing the monopole configurations. Requiring non-degeneracy of the $2k$ complex coordinates means that the values β_i and $p(\beta_i)$ be different. The discriminant of such a map defines the "resultant,

$$\Delta_k = \prod_i p(\beta_i) \neq 0 \tag{4.2}$$

which is a non-vanishing complex number.

It is of physical interest to explain the meaning of R_k. The rational map associated to a k-monopole may be found through the following. We pick a direction $\vec{u}$ in R^3 and consider

the equation $D_{\vec{u}}s = (\nabla_A - i\Phi)s = 0$ for values in R^3 orthogonal to $\vec{u}$ (parameterized by $z \in C$). We further take the parameter t to label the coordinate along $\vec{u}$ and we use a connection for ∇_A arising from the vector potential along $\vec{u}$.

We consider complex solutions to s. The solution set along the oriented line $\vec{u}$ is two dimensional and is usually denoted $E_{\vec{u}}$; the total space of all solutions defines a smooth complex vector bundle E fibred over the space of oriented lines in R^3. One now defines two sub-bundles $E^{\pm}$ with fibres $E^{\pm}_{\vec{u}}$ to be the solution space to $D_{\vec{u}}s = 0$ which decay exponentially along $\pm\vec{u}$ at infinity, respectively. The set of all $\vec{u}$ in which $E^{+}_{\vec{u}} = E^{-}_{\vec{u}}$ define a curve in the space of all oriented lines in R^3. The space of oriented lines in R^3 is isomorphic to TCP_1 and the special oriented lines are denoted spectral lines. The defining equation, the spectral curve given in equation (4.30), then defines a collection of spectral curves given by a sub-manifold in TCP_1.

The connection with the rational map is made by considering the scattering data associated with the solution space along the spectral lines. There are two linearly independent solutions s_0, s_1 as $t \to \infty$ to the operator $D_{\vec{u}}s = 0$ which have the asymptotic form given by

$$s_0(t)t^{-k/2}e^{t} \to c_0 \qquad s_1(t)t^{k/2}e^{-t} \to c_1 \ . \tag{4.3}$$

The parameters c_0 and c_1 are constants, and we see that the solution s_0 decays exponentially. Let $\tilde{s}_0$ be the corresponding solution which decays at $t \to -\infty$. Then,

$$\tilde{s}_0 = a(z)s_0(z,t) + b(z)s_1(z,t) \tag{4.4}$$

where $b(z)$ is a polynomial of at most degree k [12]. The scattering information contained in $a(z)$ and $b(z)$ define the rational map through the relation $S(z) = p(z)/b(z)$, where $p(z)$ is the unique degree $k-1$ polynomial which is $a(z)$ modulo $b(z)$.

The definition of the rational map and scattering data for the k-monopole required a preferred direction in R^3. The identification between the gauge theory vacua and the monopole moduli spaces is made more clear if we list the transformations on the map arising from:

1. SO(2) rotations about the direction $\vec{u}$,
2. global U(1) rotations corresponding to total charge,
3. translations along $\vec{u}$,
4. translations along the plane orthogonal to $\vec{u}$.

The group is $G = \mathrm{SO}(2) \otimes U(1) \otimes R \otimes C \sim SO(2) \otimes C^* \otimes C$, and we define the element by (λ, μ, ν) where $\ln|\mu| \in R$ $(\mu \neq 0)$ and $\frac{\mu}{|\mu|} \in U(1)$ and λ a phase. The total group operation induces a change in the rational map by

$$S_k(z) \to \mu^{-2}\lambda^{-2k} S_k(\frac{z-\nu}{\lambda}) \ . \tag{4.5}$$

Centering the k-monopole by fixing the center-of-mass and U(1) coordinate will leave us with a remaining SO(2) action.

We write the rational map in the form where the simple poles are manifest

$$S(z) = \sum_{i=1}^{k} \frac{\alpha_i}{z - \beta_i} \tag{4.6}$$

In the case of well-separated monopoles it is known that an approximate location for each of them is given by $\vec{x}_i$ in R^3 by

$$(\Re(\beta_i), \Im(\beta_i), -(1/2k)\ln(|p(\beta_i)|). \tag{4.7}$$

In addition, for each of the monopoles the phase associated with the unbroken U(1) is given by $p(\beta_i)/|p(\beta_i)|$. The parameters in equation (4.6) are related to the parameters in equation (4.1) by

$$\alpha_i = \frac{p(\beta_i)}{\prod_{j\neq i}(\beta_i - \beta_j)}, \tag{4.8}$$

$$b_j = (-1)^j \sum_{i_1 <,\dots,< i_j} \beta_{i_1}, \cdots, \beta_{i_j}. \tag{4.9}$$

Also if we define the vectors $A = (a_0, \dots, a_{k-1})$, $P = (p(\beta_1), \dots, p(\beta_k))$ then they satisfy the relation

$$A = BP \tag{4.10}$$

where B is the $k \times k$ Van der Monde matrix

$$B = \begin{pmatrix} 1 & \beta_1 & \beta_1^2 & \dots \\ 1 & \beta_2 & \beta_2^2 & \dots \\ \vdots & & \vdots & \ddots \end{pmatrix} \ . \tag{4.11}$$

Solving for the a_i parameters involves inverting the B matrix and is possible if and only if the $\beta_i' s$ are distinct.

The remaining transformation to discuss is (1) we list above, namely the $SO(2)$ rotation about the preferred direction. This may be found by transforming the parameters a_i and b_i as

$$a_i \to \lambda^i a_i, \qquad b_i \to \lambda^{-i} b_i. \tag{4.12}$$

It is useful to think of this transformation through an assignment of the various weights to the parameters under the rotation.

Furthermore, a natural 2-form defined on the space of the rational maps when all the coordinates β_i and $p(\beta_i)$ are distinct is given by

$$\omega = \sum_{j=1}^{k} d\beta_i \wedge d\ln p(\beta_i) \ . \tag{4.13}$$

This form is closed and is symmetric under interchange of all β_i; furthermore it is rational and defines a holomorphic symplectic form on R_k. The existence of this 2-form is directly related to the 2-form ω_h made from the three Kähler forms in the twistor description of the moduli space.

The resultant is defined by

$$\Delta(p,q) \equiv \prod_{i=1}^{k} p(\beta_i) \ , \tag{4.14}$$

which is the denominator of the $2k$-form ω^k. Physically, the center of the k-monopole is given in in eq.(4.7) and the total phase is $\arg \Delta(p,q)$. Setting $\Delta = 1$ and $b_{k-1} = 0$ corresponds to choosing the overall phase and location of the k-monopole to be zero. The $SO(2)$ action in the $1-2$ plane acts on the resultant Δ trivially. It is this action together with the weights given in eq.(4.12) we expect to maintain in taking the radius R of the compactified S^1 away from zero.

Note that the transformation (4.12) acts trivially on the resultant. One may assign a variant of this transformation which acts nontrivially on the resultant [15]

$$a_i \to \lambda^{k-i} a_i, \qquad b_i \to \lambda^i b_i, \qquad \Delta \to \lambda^{k^2} \Delta. \tag{4.15}$$

It is this transformation which is more appropriate to gauge theories and it will be related to the $U(1)_R$ symmetry. There is another symmetry that the resultant respects. It is given by a cyclic symmetry

$$a_i \to \omega a_i, \qquad \omega^k = 1. \tag{4.16}$$

Quotienting by this symmetry gives the proposed solution to the $N_f = 0$ moduli space of the gauge theory.

Let us compare the symmetries encoded in the rational map to those in the three dimensional gauge theories. The definition of a rational map requires a preferred direction in $\mathbb{R}^3$. This space is identified with the space in which the three scalars in the vector multiplet transform as a triplet. In section 2 we denoted the double of the rotation group

on this space $SU(2)_N$. Choosing a preferred direction breaks this symmetry to $SO(2)_N$. The symmetry (1) is identified with the $SO(2)$ rotations about the preferred direction parameterized by λ in the last section. The other symmetries parameterized by μ, ν which correspond to total phase and center of mass translations of the monopoles are identified with overall scale tranformation, namely setting the dynamical scale Λ to one and to the condition of tracelessness on the $SU(N)$ matrices inside $U(N)$.

Let us move to compare the parameters in the rational map to the asymptotic values of the gauge fields. We recall that asymptotically for far apart monopoles, the positions of the monopoles together with the phase of large gauge transformations are given by

$$(\Re(\beta_i), \Im(\beta_i), -(1/2k)\ln(|p(\beta_i)|, \frac{p(\beta_i)}{|p(\beta_i)|}) \tag{4.17}$$

where β_i are defined in equation (4.6) and the polynomial p is defined in equation (4.1). In view of the correspondence with gauge theories discussed in section two, we identify these as the classical expectation values of the scalar fields of the gauge theory together with the dual photon $(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, e^{i\theta_i})$, defined in equation (2.4). We see that the polynomial $q(z)$ is given asymptotically by the characteristic polynomial of the scalar field ϕ on the $1-2$ plane defind by $\phi = \phi_1 + i\phi_2$,

$$q(z) \approx \det(z - \phi),$$

where $\phi_{1,2}$ are the scalar superpartners of the vector fields in equation (2.4). Singularities of this curve correspond to points in which at least two monopoles coincide or, in the gauge theory language, an enhanced $SU(2)$ group appears. This lets us identify the coefficients b_i in equation (4.1) asymptotically as the gauge invariant symmetric functions s_i of the scalar fields ϕ, defined by Newton's formula

$$ks_k + \sum_{i=1}^{k} s_{k-i} tr(\phi^i) = 0, \qquad k = \{1, 2, \ldots\}\ , \tag{4.18}$$

where $s_0 = 1, s_1 = tr(\phi) = 0$. The physical interpretation of the coefficients a_i in equation (4.1) is slightly more complicated. Using equations (4.10) and (4.11) we find that they involve a symmetric combination of the locations on the $1-2$ plane, the phases and locations in the 3rd direction. We will give examples below. We may however see what the transformation (4.16) means. It is equivalent to a constant shift in the dual photons

$$\theta_i \to \theta_i + \frac{2\pi j}{k}, \qquad j = 1, \ldots, k. \tag{4.19}$$

Following this shift to the photons before dualizing it corresponds to an action by the center of the gauge group $\mathbb{Z}_k$. We learn that a quotient by this symmetry for monopoles is equivalent to a quotient of the gauge theory by its center.

4.1 Example: $SU(2)$ gauge theory versus two monopoles

The rational map for a centered 2 monopole and the resultant are given by

$$S(z) = \frac{a_0 + a_1 z}{z^2 + b_2}, \qquad \Delta = a_0^2 + b_2 a_1^2 = 1. \tag{4.20}$$

In [5] it was suggested that the resultant describes the curve for $SU(2)$ gauge theory with one massless flavor. The $\mathbb{Z}_2$ quotient (4.16) on the $a_i's$ is $a_{0,1} \to -a_{0,1}$. We can define invariants of this quotient by $x = a_1^2, y = a_0 a_1$ which results in a curve $y^2 + x^2 b_2 = x$. This is the curve describing the Atiyah-Hitchin manifold for two centered monopoles or, as given by [5], the curve for Yang-Mills gauge theory in three dimensions with $SU(2)$ gauge group.

We define for convenience $z_j = x_j^{(1)} + i x_j^{(2)}$, $w_j = x_j^{(3)} + i\theta_j$. They satisfy the relations for centered monopoles and zero phase $\sum_{i=1}^k z_i = \sum_{i=1}^k w_i = 0$ which are the condition to choose an $SU(k)$ gauge group from $U(k)$. Using equations (4.9) - (4.11) we find that approximately

$$\begin{aligned} a_0 &\approx \cosh w_1, & y \approx \frac{\cosh w_1 \sinh w_1}{z_1} \\ a_1 &\approx \frac{\sinh w_1}{z_1}, & x \approx \frac{\sinh^2 w_1}{z_1^2} \\ b_2 &\approx -z_1^2. \end{aligned}$$

The dots denote higher inverse powers of z_1. Recently Bielawski [19] calculated the next order in the expansion around large expectation values β_1, β_2 for two centered monopoles. The correction reads

$$|p(\beta_i)|^{\frac{1}{2}} = \frac{1+\rho}{1-\rho} e^{x_i^3}, \qquad \rho = \frac{x_1^3 - x_2^3}{|\vec{x}_1 - \vec{x}_2|} \tag{4.21}$$

Where the choice of coordinates is such that ρ is positive. This implies the form of correction from instanton contribution. It would be interesting to verify that this is indeed correct.

The connection between the charge 2 monopole and the $SU(2)$ gauge theory may be extended through the following. In this case the rational map and resultant are obtained after a "shift,"

$$S(z) = \frac{(y + xz)}{(z^2 - x^2 + u)}, \qquad \Delta = y^2 + x^2(u - x^2) \ . \tag{4.22}$$

After the Z_2 quotient with the variables chosen in equation (4.22) we obtain the elliptic curve describing the $N = 2$ theory in four dimensions with gauge group $SU(2)$ (after including the point at infinity $x, y = \infty$). We note by comparing to (4.22) that $u \approx$

$z_1^2 + \frac{\sinh^2 w_1}{z_1^2} + \ldots$. This allows us to compare to the $N = 2$ $d = 4$ relation between the period a and the global coordinates u. The curve (4.22) in the context of BPS monopoles describes two monopoles which are asymptotically located in the positions $\pm\sqrt{x^2 - u}$ in the x_1, x_2 coordinates and in the $|y \pm x\sqrt{x^2 - u}|$ in the x_3 direction. The argument of the last expression denotes the phases of the two monopoles.

A comment is in order on the scale dependence of the $d = 4$ Yang-Mills theory. The resultant in general is free to equal any complex parameter other than zero; this corresponds to uncentering the monopole. We recover the Λ dependence in the (conventionally normalized) $SU(2)$ curve by taking its value to be

$$\Delta = \frac{1}{4}\Lambda^4 , \tag{4.23}$$

On the one hand, this identifies Λ with the scale arising in the corresponding $d = 3 + 1$ theory. However, the center of mass of the monopoles is equal to $-1/4 \ln(|\Delta|)$ while the total phase of the monopole is equal to $\arg(\Delta)$. We can go over to the two centered monopole system with zero total phase by setting $\Delta = 1$. We also have the freedom to make an overall scale transformation by setting $\Lambda = 1$. This includes setting the vacuum angle θ to zero. We learn that the center of mass motion of two monopoles is identified with a scale transformation and that changing the vacuum theta angle is identified with varying the total phase of the two monopole system. This identification naturally extends to all other gauge groups.

4.2 Example: $SU(3)$ and three monopoles

For completeness we present a more general example, namely the curve proposed for the three-dimensional gauge group $SU(3)$. The rational map is

$$S(z) = \frac{a_0 + a_1 z + a_2 z^2}{z^3 + b_2 z + b_3}, \tag{4.24}$$

The resultant is

$$\Delta = a_0^3 + a_0 a_1^2 b_2 - 2a_0^2 a_2 b_2 + a_0 a_2^2 b_2^2 - a_1^3 b_3 + 3a_0 a_1 a_2 b_3 - a_1 a_2^2 b_2 b_3 + a_2^3 b_3^2 = 1 \tag{4.25}$$

Invariants of the cyclic symmetry (4.16) are given by $A_1 = a_0^2 a_1, A_2 = a_0 a_1^2, A_3 = a_0 a_1 a_2$. After the quotient we obtain the result for the curve

$$A_0^3 + A_0 A_1^2 b_2 - 2A_0^2 A_2 b_2 + A_0 A_2^2 b_2^2 - A_1^3 b_3 + 3A_0 A_1 A_2 b_3 - A_1 A_2^2 b_2 b_3 + A_2^3 b_3^2 = A_1 A_2 \ . \tag{4.26}$$

The a_i parameters take the asymptotic form

$$a_0 \approx \frac{e^{w_1} z_2 z_3}{(z_1 - z_2)(z_1 - z_3)} + \text{perm.},$$

$$
\begin{aligned}
a_1 &\approx \frac{e^{w_1}(z_2+z_3)}{(z_1-z_2)(z_1-z_3)} + \text{perm.}, \\
a_2 &\approx \frac{e^{w_1}}{(z_1-z_2)(z_1-z_3)} + \text{perm.},
\end{aligned}
$$

where the z_i parameters are defined in the previous section. We propose that the equation (4.26) describes the moduli space of three dimensional $SU(3)$ Yang-Mills theory.

4.3 BPS mass formula in three dimensions

The symplectic form defined in equation (4.13) and asymptotically in equation (2.22) serves as an integrand for calculating masses of BPS saturated states. Before stating the formula a comment is in order. Equation (4.13) gives only two of the three real symplectic forms (the real and imaginary parts). This is because in the rational maps description we are choosing a preferred direction and the forms which it calculates in a simple way are in the transverse plane to that direction. The three Kähler forms sit in the triplet of $SO(3)$ and the third may be found by a rotation in $\mathbb{R}^3$.

Recall that for the centered moduli space $\mathcal{M}_r^0$ of BPS monopoles in $SU(2)$, the fundamental group is $\pi_1(\mathcal{M}_r^0) = Z^r$. There are r non-contractible cycles in which we may integrate a one-form around; this is seen in the asymptotic form of the manifold through the presence of a T^r. A BPS mass formula for electrically charged states then reads

$$
M = |\sum_i n_i \vec{a}_i|, \qquad d\vec{a}_i = \oint_{\gamma_i} \vec{\omega}, \tag{4.27}
$$

where n_i are charges with respect to the $U(1)$ gauge fields. The set of γ_i are a convenient choice of basis of one-cycles in the $H^1(M_r^0)$. Note that the Kähler forms $\vec{\omega}$ and vectors $\vec{a}_i$ transform both as vectors under $SO(3)$.

Magnetically charged states with respect to the $U(1)$ gauge fields are instantons in three dimensions. These are the fields which contribute to the metric as discussed in section two. Their mass formula will look similar to equation (4.27) with electric charged being replaced by magnetic charges.

As an example we can calculate the semiclassical mass of the W_{ij} boson, given in equation (2.5), which has electric charges $n_i = 1$ of the gauge group $U(1)_i$ and $n_j = -1$ of the gauge group $U(1)_j$ and $n_k = 0$ in the rest. We choose the basis of one cycles γ_i to satisfy the relation $\oint_{\gamma_i} d\tilde{\theta}_j = \delta_{ij}$. Then we have asymptotically, using the approximate Kähler forms in equation (2.22),

$$
d\vec{a}_i = \oint_{\gamma_j} (d\tilde{\theta}_i \wedge d\vec{x}_i), \tag{4.28}
$$

where the integration over the first term in equation (2.22) vanishes. This calculation identifies asymptotically $\vec{a}_i$ with $\vec{x}_i$. This occurs after matching the parameters a_i and x_i in the asymptotic regime and with the appropriate normalization in equation (4.28). We obtain the expected relation to the mass formula, using (4.27),

$$M_{ij} = |\vec{x}_i - \vec{x}_j|. \tag{4.29}$$

Equation (4.27) also serves as the BPS formula for a tension of a string in four dimensions. This follows from the correspondence between strings in four dimensions and particles in three dimensions as discussed in the introduction. When the BPS expression vanishes tensionless string arise in the spectrum.

4.4 Comments on Spectral Curves

The $R \to \infty$ limit in principle should reduce to the hyper-elliptic curves associated to $d = 3+1$ $N = 2$ super Yang-Mills theory, after including the appropriate points at infinity to obtain a compact surface. There is another point of view associated with the equivalent spectral curve description of monopoles; we refer the reader to the references [12] for a complete description of the twistor construction.

Associated to the charge k monopole moduli space is a spectral curve in TCP_1, the tangent bundle over CP_1,

$$\eta^n + a_1(\xi)\eta^{n-1} + \ldots + a_n(\xi) = 0 \ . \tag{4.30}$$

The local coordinates in TCP_1 of the curve are given by $\rho\frac{d}{d\xi}$. For strongly centered monopoles the first coefficient $a_1(\xi)$ is zero.

Holomorphic polynomials of order $2n$ over CP_1 are defined by

$$a_n(\xi) = \sum_{j=0}^{2n} w_j \xi^j \ . \tag{4.31}$$

The functions $a_n(\xi)$ are sections of line bundles $\mathcal{O}(2n)$ over CP_1 consisting of all polynomials of order $2n$, and ξ is the CP_1 coordinate. Furthermore, the standard representation of CP_1 is found by pasting together two copies of the complex plane Φ, $\tilde{\Phi}$ with coordinates ξ and $\tilde{\xi}$; on the overlap $\Phi \cap \tilde{\Phi}$ the coordinates are related by $\xi = 1/\tilde{\xi}$. On the overlap of the two charts the functions in equation(4.31) transform as $\tilde{a}_n(1/\xi) = \xi^{-2n} a_n(\xi)$.

In this description a reality constraint on a_n given by

$$\bar{a}_n(\xi) = (-1)^n \bar{\xi}^{2n} a_n(-1/\bar{\xi}) \ , \tag{4.32}$$

which means that $w_j = (-)^{n+j}\bar{w}_{2n-j}$. The operation in equation (4.32) may be regarded as invariance under complex conjugation together with the anti-podal map; in this case the real structure is a map which takes CP_1 onto itself defined on the above functions as $a_n(\xi) \to \bar{a}_n(-1/\bar{\xi})$.

There is an additional map which acts on the CP$_1$ coordinates as $\xi \to -1/\bar{\xi}$ and $\eta \to -\frac{\bar{\eta}}{\xi^2}$. This is the analog of complex conjugation on the twistor space and takes the complex structure on the twistor space to its inverse (i.e. complex conjugation $I \to -I$). Demanding compatibility of the curve with the real structure enforces the coordinates $a_n(\xi)$ to satisfy the reality constraint in equation (4.32).

We now turn to the relation of monopoles moduli spaces to $N = 4$ Yang-Mills theories in three dimensions. We proposed that $SU(n)$ Yang-Mills theories have moduli space of n monopoles, obtained by quotienting the rational map by a cyclic group corresponding in the gauge theory description to the center. This result has a natural generalization to ADE groups which is related to the discrete point groups of solids. For D_n series we propose that $SO(2n)$ Yang-Mills theories are obtained by quotienting the rational map of $2n$ monopoles by the dihedral group (generated by the cyclic generator and an inversion). For the E_n series a similar relation is proposed where the discrete group is appropriate to the E_n group.

We would like to point out the following interesting procedure which does not seem to agree with our proposal. First recall the result of [5] for gauge group $SU(2)$. The moduli space for $N_f = 0$ was identified as the Atiyah-Hitchin space. One way to construct this space is to consider its simply connected double cover and then mod by the cyclic symmetry of two monopoles - Z_2. We will now describe a procedure due to Sutcliffe [16] which relates the spectral curves to the curves associated with $d = 4$ super Yang-Mills theory. We recall the spectral curve for two monopoles

$$\eta^2 + a_1(\xi)\eta + a_2(\xi) = 0 \tag{4.33}$$

The SO(3) spatial rotations of the underlying Euclidean space R^3 induce actions on the TP_1 coordinates (given for example in [14]). The various quotients on the spectral curve we are describing translate directly into symmetric configurations of a multi-monopole. The Z_2 acts on the TP_1 coordinates as

$$\eta \to -\eta \qquad \xi \to -\xi \tag{4.34}$$

Imposing this symmetry and the reality condition equation (4.32) we get a curve for the $SU(2)$ gauge theory in four dimensions

$$\eta^2 + \xi^2 u_2 + \beta\xi^4 + \bar{\beta} = 0. \tag{4.35}$$

We mod by the symmetry (4.34) which has invariants $x = \frac{\eta}{\xi}$ and $z = \xi^2\beta$, set $\mu = |\beta|^2$ and get

$$z + \frac{\mu}{z} + x^2 + u = 0. \tag{4.36}$$

This is the curve for $SU(2)$ Yang-Mills theory as written in [13]. The above is easily generalized to $SU(N)$ gauge theories and we refer to [16] for the details. Moreover we have performed a generalization of this to D_n series. Taking the spectral curve for $2n$ monopoles and moding by the symmetry

$$(\xi, \eta) \to (\omega\xi, \omega\eta), \qquad (\xi, \eta) \to (-\frac{\eta}{\xi^2}, \frac{1}{\xi}), \tag{4.37}$$

We get the curve for D_n series as written in [13]. We expect the results to be generalized to E_n groups in a similar way.

The above construction is highly suggestive to make the connection with the hyper-elliptic curves of $N = 2$ super Yang-Mills in four dimensions. However, the coefficients in the curve are imposed to satisfy the reality constraint in eq.(4.32) which implies that the coefficients in the quotient of the curve are real; however, for gauge theories the parameters labeling the moduli space are complex. The connection as a result is unclear.

5 Discussion

The conjectured relation between $N = 4$ low-energy effective actions and the moduli space of BPS solitons raises many interesting questions. Besides expectations based on string theory, we would first like to point out that the asymptotic boundary conditions on the metric together with the appropriate $SO(3)$ action imposes a strong constraint on the correspondence between the gauge theory and monopole moduli spaces which needs to be investigated further.

One would like to verify that instanton corrections in the three dimensional theory actually reproduce the exponential corrections to the asymptotic form of the metric, which are alternatively known to arise in the monopole picture through exponentially damped charge exchange processes in the slow dynamics of magnetic monopoles. For example, in $SU(2)$ with $N_f = 0$ hypermultiplets these instanton corrections will exactly reproduce the Atiyah-Hitchin metric. In this case, this statement is exact for the reason that the only 4-dimensional hyper-Kähler metric in which one may write down that possesses the correct $SO(3)$ action gives rise to both the moduli space of $d = 3$ vacua and the centered moduli space of a charge 2 BPS monopole. The correspondence for $SU(2)$ with $N_f = 0$ predicts a

non-trivial structure for the instanton corrections in the three-dimensional gauge theory; we suggest the same for higher-rank gauge groups.

Further, whereas the dynamics of magnetic monopoles is modelled by a first quantized $N = 4$ supersymmetric theory, the three dimensional gauge theory is a second-quantized field theory. The connection between the two approaches is unresolved. A possible explanation may arise as follows. All charge k-monopole moduli space metrics come from the inner product of the zero mode field deformations to the solutions of the BPS equations of a charge k field solution in $SU(2) \rightarrow U(1)$. From the $2+1$ dimensional gauge theory point of view the previous statement implies the existence of a master set of Bogolmony equations governing a space of theories with zero mode deformations responsible for the array of three dimensional nonperturbative $SU(N_c)$ $N = 4$ super Yang-Mills theories. The zero modes about the master field equation, acting on a functional with appropriate boundary conditions as in the monopole problem, would give rise to the gauge theories of different rank. In this sense, an approach based on third quantization would explain how all the $SU(N_c)$ theories in $2+1$ dimensions with $N_f = 0$ are related. It would be interesting to explore this relation in more detail including its connection to the nonperturbative results of $N = 2$ super Yang-Mills theories in four dimensions (i.e. taking the radius of the S^1 to infinity from the three-dimensional point of view).

The motivation to study three dimensional gauge theories came from string theory as outlined in the introduction. Simple configurations in string theory contain more matter multiplets in the fundamental adjoint and various symmetric representations. For example a configuration of intersecting 5-branes in M-theory will be associated with the adjoint representation of $U(n)$ where n is the charge of one of the five branes. Such a gauge theory has a negative s parameter and therefore the semi-classical result is expected to be exact. In particular no instanton like corrections are expected. However this suggests a "one loop" correction to the classical solution in M-theory. It would be interesting to check this. Other applications of monopoles to string theory can be found by associating Manton scattering to five brane scattering in M-theory.

We conclude with several comments. The appearance of closed geodesics appearing in the monopole moduli spaces should have an interesting analogy for the gauge theory. The correspondence in the latter is plausibly related with the marginally unstable states occuring in the BPS spectrum.

Finally we remark on a possible theory of "nonabelian" linear multiplets which has part of its moduli space the monopole moduli space. At points where the BPS mass formula vanishes we expect to get tensionless strings in four dimensions.

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